Mathematical progress in general relativity

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Introduction

This talk concerns

- mathematics;
- Einstein's equations;
- the Cauchy problem;
- singularities;
- the cosmic censorship conjectures.

Classical general relativity

- The Einstein equations in $(3+1)$-dimensions:

$$Ric_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = -\Lambda g_{\mu\nu} + 2T_{\mu\nu}.$$

- Assuming $\Lambda = 0$ and vacuum ($T_{\mu\nu} = 0$), this reduces to

$$Ric_{\mu\nu}(g) = 0.$$

- This could be viewed as a system of nonlinear wave equations.

Cauchy problem in general relativity

Theorem (Choquet-Bruhat–Geroch (1969))

Given sufficiently regular data, there exists a unique maximal globally hyperbolic future development $(\mathcal{M}, g)$ solving the Einstein vacuum equations.

- Matter fields can be added in the initial value problem.
- This also allows us to consider dynamical questions such as stability, genericity, etc.
- We are mostly interested in the asymptotically flat setting.

Black hole solutions

- The Kerr family ($0 \leq |a| \leq M$) (1963) is expected to play a fundamental role in the dynamics of the Einstein equations!

Conjecture (No hair conjecture)

Kerr is the only smooth stationary asymptotically flat vacuum black hole.

- Carter (1971), Robinson (1975), Hawking, Alexakis–Ionescu–Klainerman (2010), etc.
- Suggests that "Kerr is the generic end-state of evolution".

The Schwarzschild spacetime

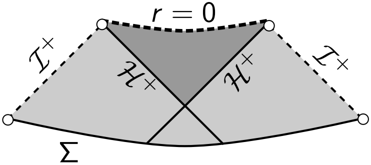


Figure: Maximal globally hyperbolic future development of Schwarzschild

Stability of Kerr exterior conjecture

Conjecture (Stability of Kerr exterior)

Given an initial data set to the Einstein vacuum equation which is globally close to Kerr initial data with $0 \leq |a| < M$*. The maximal globally hyperbolic development has an exterior region which converges to a nearby Kerr spacetime.*

- Lots of progress! (Regge–Wheeler, Teukolsky, Chandrasekhar, Whiting, . . . , Alinhac, Anderson, Aretakis, Blue, Bony, Dafermos, Dyatlov, Finster, Häfner, Hintz, Holzegel, Hung, Ionescu, Kamran, Kay, Keller, Lindblad, Ma, Marzuola, Metcalfe, Pasqualotto, Rodnianski, Shlapentokh-Rothman, Schlue, Smoller, Smulevici, Soffer, Sterbenz, Szeftel, Tataru, Taylor, Tohaneanu, Vasy, Wald, Wang, Warnick, Wunsch, Yau, Zworski, . . .)

Stability of Kerr exterior conjecture

Theorem (Dafermos–Holzegel–Rodnianski–Taylor (in progress), Klainerman–Szeftel (2017))

The Schwarzschild exterior is globally nonlinearly stable (in some appropriate restricted class).

Theorem (Hintz–Vasy (2016))

Slowly rotating Kerr–de Sitter exteriors are globally nonlinearly stable.

The Schwarzschild singularity

- Sitting inside the Schwarzschild black hole ($a = 0$, $M > 0$) is a singularity.
- This is "singular" and can be seen in three ways:
  - Geodesically incomplete.
  - Curvature invariant $R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$ blows up.
    - C^2 inextendibility of metric
  - Observers are infinitely torn apart
    - related to C^0 inextendibility of metric

Theorem (Penrose (1965))

Let $(\mathcal{M}, g)$ be the maximal globally hyperbolic future development of asymptotically flat data. If $(\mathcal{M}, g)$ contains a trapped surface, then it is geodesically incomplete.

The Schwarzschild singularity

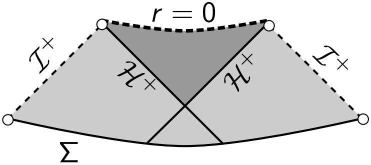


Figure: Maximal globally hyperbolic future development of Schwarzschild

Cosmic censorship conjectures

Its singularity notwithstanding, Schwarzschild has the following preferable features:

- The spacetime cannot be extended as a larger solution to Einstein equations, and therefore does not cause determinism to break down.
- The singularity is invisible to far-away observers.

The cosmic censorship conjectures of Penrose assert that these preferable features of the Schwarzschild are generic for asymptotically flat vacuum solutions.

Strong cosmic censorship conjecture

A mathematical formulation of the first expectation is the following

Conjecture (Strong cosmic censorship, Penrose)

Maximal globally hyperbolic future developments to generic asymptotically flat initial data are future inextendible as suitably regular Lorentzian manifold.

"Suitably regular" can be made more precise:

Conjecture (C^k formulation of strong cosmic censorship)

Maximal globally hyperbolic future developments to generic asymptotically flat initial data are future inextendible as a Lorentzian manifold with a C^k metric.

- C^2 is related to curvature blowup; C^0 is related to infinite tidal deformation.

Spacelike singularity conjecture

The strong cosmic censorship conjecture is often associated with "spacelike singularities".

Conjecture (Spacelike singularity conjecture)

Generic asymptotically flat initial data have a maximal globally hyperbolic future development whose "finite" future boundary is everywhere spacelike.

- More ambitiously, "BKL picture", etc.

Weak cosmic censorship conjecture

A mathematical formulation of the expectation the “singularities are invisible to far-away observers” is the following

Conjecture (Weak cosmic censorship conjecture)

Maximal globally hyperbolic future developments to generic asymptotically flat initial data possess complete null infinities.

The Kerr spacetime

- Ironically, even the cosmic censorship conjectures could be motivated by Schwarzschild, the Schwarzschild singularity is quite special!
- Indeed, stable spacelike singularities are only known with matter fields or in high dimensions ≥ 37 (Rodnianski–Speck).
- When $0 < |a| < M$, Kerr (or when $0 < |Q| < M$, Reissner–Nordström) is geodesically incomplete but has no singularities.
- In fact, infinite number of extensions solve the Einstein vacuum equations!

The Kerr spacetime

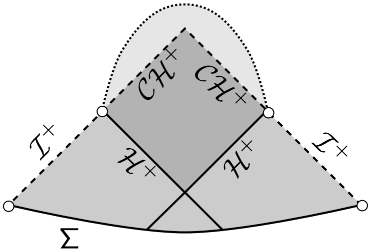


Figure: Maximal globally hyperbolic future development of Kerr and a non-unique extension

Strong cosmic censorship conjecture

- According to the strong cosmic censorship conjecture, the smooth Cauchy horizon of Kerr is expected to be unstable!
- The exterior stability conjecture says nothing about the interior.
- Instability is thought to be driven by a blue-shift mechanism.
- Extensively studied in the physics literature (Penrose, Penrose–Simpson, McNamara, Gürsel–Sandberg–Novikov–Starobinsky, Chandrasekhar–Hartle, Hiscock, Poisson–Israel, Ori, . . .)

Toy model

Einstein–Maxwell–scalar field system with spherically symmetric initial data.

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2(T^{(sf)}_{\mu\nu} + T^{(em)}_{\mu\nu}), \\ T^{(sf)}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \\ T^{(em)}_{\mu\nu} = (g^{-1})^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} (g^{-1})^{\alpha\beta} (g^{-1})^{\gamma\sigma} F_{\alpha\gamma} F_{\beta\sigma}, \end{cases}$$

where F is a 2-form and ϕ is a real-valued function satisfying

$$\Box_g \phi = 0, \quad dF = 0, \quad (g^{-1})^{\alpha\mu} \nabla_\alpha F_{\mu\nu} = 0.$$

C^0-formulation of strong cosmic censorship is false!

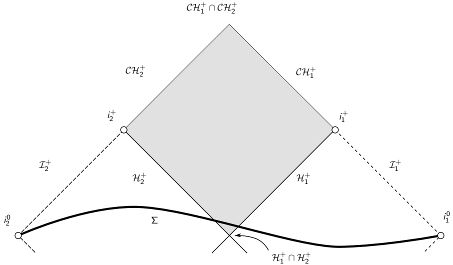
Theorem (Dafermos, Dafermos–Rodnianski (2005))

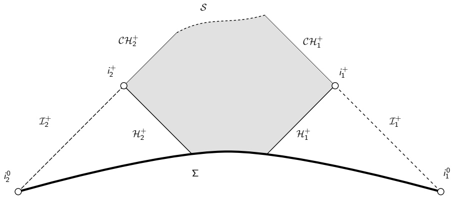
Consider spherically symmetric, 2-ended asymptotically flat admissible initial data to the Einstein–Maxwell–scalar field system with non-trivial Maxwell field. Then the maximal globally hyperbolic future development

- *is C^0-future-extendible; and*
- *and its finite boundary has a null component.*

Both the C^0 formulation of strong cosmic censorship conjecture and the spacelike singularity conjecture are **false**!

Solutions arising from spherically symmetric data





C^2 formulation of strong cosmic censorship in spherical symmetry

The extensions in general cannot be classical solutions to the Einstein–Maxwell–scalar field system.

Theorem (L.-Oh (2017))

There exists a generic (open and dense) set $\mathcal{G}$ of spherically symmetric two-ended asymptotically flat admissible smooth initial data to the Einstein–Maxwell–scalar field system such that the maximal globally hyperbolic future development is C^2-future-inextendible.

- In particular, the Reissner–Nordström Cauchy horizon is C^2-unstable!

Vacuum equations without symmetry assumptions

We turn to the **vacuum** problem **with no symmetry assumptions**.

Stability of the Kerr Cauchy horizon

We **assume** the validity of the stability of Kerr exterior conjecture.

Theorem (Dafermos–L. (2017))

If the stability of Kerr exterior conjecture is true (with quantitative decay rates), then

- *the maximal globally hyperbolic future development to any small perturbation of Kerr data (with $0 < |a| < M$) has the Penrose diagram of Kerr.*
- *Moreover, the metric is continuously extendible to the Cauchy horizon, and*
- *(in appropriate coordinate systems) is C^0-close to the Kerr metric.*

Stability of the Kerr Cauchy horizon

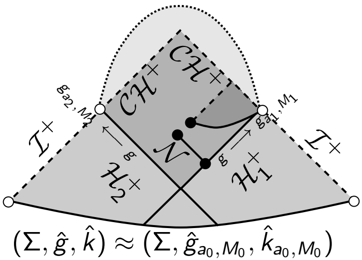


Figure: The global stability of the Kerr Penrose diagram

C^0 formulation of strong cosmic censorship is false!

As a consequence,

Corollary

If the stability of Kerr exterior conjecture is true, then the C^0 formulation of the strong cosmic censorship conjecture is **false**.

Corollary

If the stability of Kerr exterior conjecture is true, then the spacelike singularity conjecture is **false**.

Instability conjecture

While we showed that the Cauchy horizon is C^0 stable, our proof suggests that higher derivatives of the metric may blow up:

Conjecture

For generic perturbations as in the theorem before, the spacetime cannot be extended as a C^2-Lorentzian manifold.

Stability of the Kerr Cauchy horizon

Assuming the event horizon approaches the Kerr event horizon (with $0 < |a| < M$) sufficiently fast, then the interior of the black hole has "a nonempty piece of Cauchy horizon near timelike infinity".

- This holds even if the data is not close to Kerr everywhere!

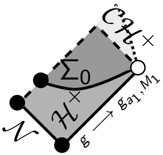


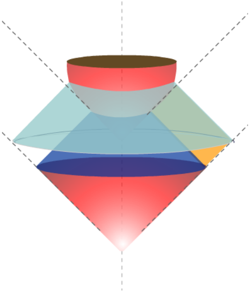
Figure: Stability of $\mathcal{CH}^+$ from data on $\mathcal{H}^+ \cup \mathcal{N}$

Formation of trapped surface in vacuum

- How do black holes arise dynamically from "reasonable" data?
- Do trapped surface form dynamically from dispersed data?

Theorem (Christodoulou (2008))

Trapped surfaces may form dynamically from arbitrarily dispersed initial data via the focusing of sufficiently strong gravitational waves.



Formation of trapped surface in vacuum

In fact, there is a much larger set of initial configuration that leads to trapped surface formation:

Theorem (Klainerman–L.–Rodnianski (2014))

Trapped surfaces may form dynamically from arbitrarily dispersed initial data via the focusing of sufficiently strong gravitational waves **concentrated only in one direction**.

Aside: Instability of anti–de Sitter

- In the asymptotically flat setting, trapped surfaces can be formed by focusing of strong gravitational waves.
- In the asymptotically anti-de Sitter (AdS) case, the confinement allows for additional amplification.

Conjecture (Dafermos–Holzegel (2006))

AdS is unstable for the Einstein vacuum equation with negative cosmological constant with reflecting boundary conditions. In particular, there exists arbitrarily small initial data which lead to the formation of trapped surfaces.

- Many numerical and heuristic works starting with Bizon–Rostworowski (2011).

Instability of anti–de Sitter

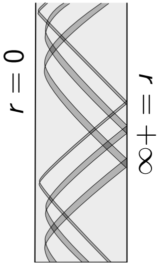


Figure: Instability of AdS

Theorem (Moschidis (2018))

AdS is unstable for the Einstein–massless Vlasov system with reflecting boundary conditions in spherical symmetry. In particular, there exists arbitrarily small initial data which lead to the formation of trapped surfaces.

Naked singularities

- Assuming the stability of Kerr exterior conjecture, **weak cosmic censorship conjecture holds in a neighborhood of Kerr**.
- Yet for data far away from Kerr it is in principle possible to have a naked singularity.

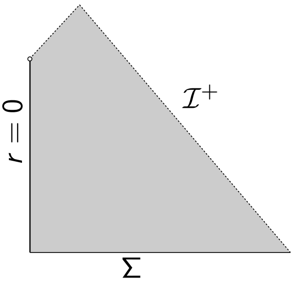


Figure: Naked singularity

Naked singularities

Theorem (Christodoulou (1994))

Naked singularities exist in spherical symmetry for the Einstein–scalar field model.

- Constructed as a self-similar solution.
- Numerical results (Choptuik (1993)) suggests a different naked singularity which arises from smooth data.

Naked singularities

Conjecture

Naked singularities exist in vacuum.

- A theory of self-similar solutions — which does not incorporate naked singularities — has very recently been developed by Rodnianski–Shlapentokh-Rothman (2018).

Weak cosmic censorship in spherical symmetry

Nevertheless,

Theorem (Christodoulou (1999))

The weak cosmic censorship conjecture holds in spherical symmetry for the Einstein–scalar field model.

- Given any naked singularity, there exist arbitrarily small perturbations so that the singularity is hidden behind an event horizon.
- The fundamental insight is that a naked singularities provides an infinite blue-shift for small perturbations, which then lead to trapped surface formation.

Trapped surface conjecture

Christodoulou's theorem in spherical symmetry suggests an avenue to attack the weak cosmic censorship conjecture via proving the following:

Conjecture (Trapped surface conjecture)

Naked singularities are unstable to trapped surface formation.

Conjecture

The trapped surface conjecture implies the weak cosmic censorship conjecture.

Perhaps the theorems on trapped surface formation in vacuum may eventually be relevant (cf. An–L. (2014), Li–Liu (2017))?

Thank you!